

ON JETS, EXTENSIONS AND CHARACTERISTIC CLASSES II

HELGE MAAKESTAD

ABSTRACT. In this paper we define the generalized Atiyah classes $c_{\mathcal{J}}(\mathcal{E})$ and $c_{\mathcal{O}_X}(\mathcal{E})$ of a quasi coherent sheaf \mathcal{E} with respect to a pair (\mathcal{I}, d) where \mathcal{I} is a left and right \mathcal{O}_X -module and d a derivation. We relate this class to the structure of left and right module on the first order jet bundle $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$. In the main Theorem of the paper we show $c_{\mathcal{O}_X}(\mathcal{E}) = 0$ if and only if there is an isomorphism $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{left} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{right}$ as \mathcal{O}_X -modules. We also give explicit examples where $c_{\mathcal{O}_X}(\mathcal{E}) \neq 0$ using jet bundles of line bundles on the projective line hence the classes $c_{\mathcal{J}}(\mathcal{E})$ and $c_{\mathcal{O}_X}(\mathcal{E})$ are non trivial. The classes we introduce generalize the classical Atiyah class.

CONTENTS

1. Introduction	1
2. Generalized Atiyah classes for modules over rings	2
3. Generalized Atiyah classes for quasi coherent sheaves	8
4. Jets and infinitesimal extensions of sheaves	13
5. Appendix: Explicit examples	14
References	19

1. INTRODUCTION

The aim of this paper is to introduce and study generalized first order jet bundles and generalized Atiyah classes for quasi coherent sheaves relative to an arbitrary morphism $\pi : X \rightarrow S$ of schemes. We define the generalized first order jet bundle $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ of \mathcal{E} and the generalized Atiyah sequence

$$(1.0.1) \quad 0 \rightarrow \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow \mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow 0.$$

The generalized Atiyah sequence is an exact sequence of quasi coherent sheaves of left and right \mathcal{O}_X -modules and left and right $\mathcal{J}_{\mathcal{I}}^1$ -modules. The sheaf $\mathcal{J}_{\mathcal{I}}^1$ is a sheaf of associative rings on X . It is an extension of \mathcal{O}_X with a left and right quasi coherent \mathcal{O}_X -module \mathcal{I} of square zero. The main result of the paper is that the sequence 1.0.1 is left split as sequence of \mathcal{O}_X -modules if and only if the left and right structure on $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ are \mathcal{O}_X -isomorphic (see Theorem 3.6).

We give a general definition of the first order jet bundle $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ of a quasi coherent sheaf \mathcal{E} using a derivation d and a left and right \mathcal{O}_X -module \mathcal{I} . We define generalized Atiyah classes $c_{\mathcal{J}}(\mathcal{E})$ and $c_{\mathcal{O}_X}(\mathcal{E})$ of \mathcal{E} and relate these classes to the left and right

Date: Spring 2009.

1991 Mathematics Subject Classification. 14F10, 14F40.

Key words and phrases. Atiyah sequence, jet bundle, characteristic class, generalized Atiyah class, square zero extension, lifting.

\mathcal{O}_X structure on $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$. The generalized Atiyah class $c_{\mathcal{O}_X}(\mathcal{E})$ measures when there is an isomorphism

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{left} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{right}$$

as \mathcal{O}_X -modules (see Theorem 3.6). There is always an isomorphism

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{left} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{right}$$

as sheaves of abelian groups. Hence the class $c_{\mathcal{O}_X}(\mathcal{E})$ measures when the sheaf of abelian groups $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ may be given two non isomorphic structures as \mathcal{O}_X -module. When $\mathcal{I} = \Omega_{X/S}^1$ and $d : \mathcal{O}_X \rightarrow \Omega_{X/S}^1$ is the universal derivation it follows the characteristic class $c_{\mathcal{O}_X}(\mathcal{E})$ is the classical Atiyah class as defined in [2].

We prove in Example 3.10 there is no isomorphism

$$\mathcal{J}_{\mathbf{P}^1}^1(\mathcal{O}(d))^{left} \neq \mathcal{J}_{\mathbf{P}^1}^1(\mathcal{O}(d))^{right}$$

of $\mathcal{O}_{\mathbf{P}^1}$ -modules on \mathbf{P}^1 . Hence the class $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d))$ is non-zero for the sheaf $\mathcal{O}(d)$ on the projective line \mathbf{P}^1 over a field K of characteristic zero. If $\text{char}(K)$ divides d it follows $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d))$ is zero hence in this case there is an isomorphism

$$\mathcal{J}_{\mathbf{P}^1}^1(\mathcal{O}(d))^{left} \cong \mathcal{J}_{\mathbf{P}^1}^1(\mathcal{O}(d))^{right}$$

of $\mathcal{O}_{\mathbf{P}^1}$ -modules. It follows the generalized Atiyah classes we introduce are non trivial.

In a previous paper on jet bundles (see [10]) we found many examples of jets of line bundles on the projective line where the structure as left locally free $\mathcal{O}_{\mathbf{P}^1}$ -module was nonisomorphic to the structure as right locally free $\mathcal{O}_{\mathbf{P}^1}$ -module. In this paper we interpret this phenomenon in terms of the generalized Atiyah class.

In the first section of the paper we do all constructions in the local case. In the second part of the paper we do all constructions in the global case. In the final section of the paper we discuss Atiyah sequences and Atiyah classes relative to arbitrary morphisms of schemes and arbitrary infinitesimal extensions of \mathcal{O}_X by a quasi coherent left and right \mathcal{O}_X -module. We get a definition of Atiyah classes $c_{\mathcal{J}}(\mathcal{E})$ and $c_{\mathcal{O}_X}(\mathcal{E})$ for an arbitrary quasi coherent sheaf \mathcal{E} of \mathcal{O}_X -modules. Here \mathcal{J} is an infinitesimal extension of \mathcal{O}_X with respect to a quasi coherent left and right \mathcal{O}_X -module \mathcal{I} . The construction generalize the classical case (see Example 4.5). In the final section of the paper we construct explicit examples. We also construct (see Example 5.6) for any smooth projective scheme X over the complex numbers a characteristic class $c_{\mathcal{I}}(\mathcal{E})$ in $K_0(X)$ where $K_0(X)$ is the Grothendieck group of locally free finite rank sheaves on X . There is a canonical map of Grothendieck groups

$$\gamma : K_0(X) \rightarrow KO(X(\mathbf{R}))$$

where $X(\mathbf{R})$ is the underlying real smooth manifold of X and $KO(X(\mathbf{R}))$ is the Grothendieck group of the category of real finite rank smooth vector bundles on $X(\mathbf{R})$. The class $c_{\mathcal{I}}(\mathcal{E})$ lies in the group $\ker(\gamma)$ for any locally free finite rank sheaf \mathcal{E} and any left and right \mathcal{O}_X -module \mathcal{I} .

2. GENERALIZED ATIYAH CLASSES FOR MODULES OVER RINGS

Let in the following $\phi : A \rightarrow B$ be a unital morphism of commutative rings. Let E be a B -module. Let I be a left and right B -module with $a(xb) = (ax)b$ for all $a, b \in B$ and $x \in I$ and let $d \in \text{Der}_A(B, I)$ be a derivation. This means $d(\phi(a)) = 0$

for all $a \in A$ and $d(ab) = ad(b) + d(a)b$ for all $a, b \in B$. In the following we will write $d(a)$ instead of $d(\phi(a))$.

Let $\mathcal{J}_I^1 = I \oplus B$ with the following multiplication:

$$(x, a)(y, b) = (xb + ay, ab).$$

It follows \mathcal{J}_I^1 is an associative ring with multiplicative unit $\mathbf{1} = (0, 1)$. Define a map $d_I : B \rightarrow \mathcal{J}_I^1$ by $d_I(b) = (d(b), 0)$. Define maps $s, t : B \rightarrow \mathcal{J}_I^1$ by

$$t(b) = (0, b)$$

and

$$s(b) = (d(b), b).$$

Define two left actions of B on \mathcal{J}_I^1 as follows:

$$(2.0.2) \quad (x, a)_t b = (x, a)t(b) = (xb, ab) \text{ and } b_t(x, a) = t(b)(x, a) = (bx, ba)$$

and

$$(2.0.3) \quad (x, a)_s b = (x, a)s(b) = (xb + ad(b), ab)$$

$$(2.0.4) \quad b_s(x, a) = s(b)(x, a) = (d(b)a + bx, ba).$$

The ring \mathcal{J}_I^1 acts canonically on $I \subseteq \mathcal{J}_I^1$ as follows:

$$(x, 0)(y, b) = (xb, 0) \text{ and } (y, b)(x, 0) = (bx, 0).$$

In the following when we write $\text{Der}_A^t(B, \mathcal{J}_I^1)$ we mean A -linear derivations with B -structure on \mathcal{J}_I^1 induced by t from 2.0.2.

Proposition 2.1. *The following holds:*

$$(2.1.1) \quad \mathcal{J}_I^1 \text{ is a square zero extension of } B \text{ by } I.$$

$$(2.1.2) \quad 2.0.2\text{-}2.0.4 \text{ define } \mathcal{J}_I^1 \text{ as left and right } B\text{-module.}$$

$$(2.1.3) \quad \text{The map } d_I \text{ is a derivation: } d_I \in \text{Der}_A^t(B, \mathcal{J}_I^1).$$

$$(2.1.4) \quad \text{The maps } s, t \text{ are ring homomorphisms and } s - t = d_I.$$

Proof. The proof is left to the reader. \square

Define the following map:

$$d_{\mathcal{J}} : \mathcal{J}_I^1 \rightarrow I$$

by

$$d_{\mathcal{J}}(x, a) = x + d(a).$$

When we write $\text{Der}_A^t(\mathcal{J}_I^1, I)$ we mean derivations linear over A with respect to the left action of A on \mathcal{J}_I^1 induced by t from 2.0.2.

Lemma 2.2. *The following holds: $d_{\mathcal{J}} \in \text{Der}_A^t(\mathcal{J}_I^1, I)$.*

Proof. The proof is left to the reader. \square

Define the following abelian group:

$$\mathcal{J}_I^1(E) = \mathcal{J}_I^1 \otimes_B E \cong I \otimes_B E \oplus E.$$

Define the following left and right action of \mathcal{J}_I^1 on $\mathcal{J}_I^1(E)$:

$$(2.2.1) \quad (x, a)(z \otimes e, f) = (x \otimes f + az \otimes e + d(a) \otimes f, af)$$

and

$$(2.2.2) \quad (z \otimes e, f)(x, a) = (z \otimes ea, fa).$$

Note: We may write

$$(x, a)(z \otimes e, f) = ((az) \otimes e + d_{\mathcal{J}}(x, a) \otimes f, af).$$

Definition 2.3. Let $\mathcal{J}_I^1(E)$ with the actions 2.2.1 and 2.2.2 be the *first order I -jet module* of E with respect to d .

There is an exact sequence of abelian groups

$$(2.3.1) \quad 0 \rightarrow I \otimes_B E \rightarrow \mathcal{J}_I^1(E) \rightarrow E \rightarrow 0$$

There is a left and right action of \mathcal{J}_I^1 on E defined as follows:

$$(2.3.2) \quad (x, a)e = ae$$

$$(2.3.3) \quad e(x, a) = ea$$

Proposition 2.4. *The actions 2.2.1 and 2.2.2 make $\mathcal{J}_I^1(E)$ into a left and right \mathcal{J}_I^1 -module. The exact sequence 2.3.1 is an exact sequence of left and right \mathcal{J}_I^1 -modules.*

Proof. The proof is left to the reader. \square

Corollary 2.5. *The sequence 2.3.1 is an exact sequence of left and right B -modules. It is split exact as right B -modules.*

Proof. Use the ring homomorphism $t : B \rightarrow \mathcal{J}_I^1$ defined by $t(b) = (0, b)$. It follows the maps in the sequence 2.3.1 are B -linear. The Corollary now follows from Proposition 2.4. \square

Definition 2.6. Let the sequence 2.3.1 be the *Atiyah-Karoubi sequence* of E with respect to the pair $(I, d_{\mathcal{J}})$.

Note: The exact sequence 2.3.1 was first defined in [2] in the case when $\mathcal{I} = \Omega_{X/\mathbf{C}}$, d is the universal derivation and X is a complex manifold.

Recall the following definition: Given two left B -modules E, F we define

$$\text{Diff}_A^0(E, F) = \text{Hom}_B(E, F)$$

and

$$\text{Diff}_A^k(E, F) = \{\partial \in \text{Hom}_A(E, F) : [\partial, a] \in \text{Diff}_A^{k-1}(E, F) \text{ for all } a \in B\}.$$

There is a natural map

$$d_E : E \rightarrow \mathcal{J}_I^1(E)$$

defined by

$$d_E(e) = (0, 1) \otimes e$$

called the *universal differential operator of E* with respect to the pair (I, d) . Here $(0, 1) = \mathbf{1} \in \mathcal{J}_I^1 = I \oplus B$ is the multiplicative unit.

Lemma 2.7. *It follows $d_E \in \text{Diff}_A^1(E, \mathcal{J}_I^1(E))$.*

Proof. Recall the following: For elements $(x, a) \in \mathcal{J}_I^1$ and $b \in B$ it follows

$$(x, a)_t b = (x, a)(0, b) = (xb, ab)$$

and

$$b_s(x, a) = (d(b), b)(x, a) = (bx + d(b)a, ba).$$

It follows $d_E \in \text{Diff}^1(E, \mathcal{J}_I^1(E))$ if and only if $[d_E, a] \in \text{Hom}_B(E, \mathcal{J}_I^1(E))$ for all $a \in B$. We get

$$\begin{aligned} [d_E, a](be) &= (d_E a - ad_E)(be) = d_E(abe) - ad_E(be) = (0, 1) \otimes abe - a(0, 1) \otimes be = \\ &= (0, ab) \otimes e - a(0, 1) \otimes be = (0, ab) \otimes e - (d(a)b, ab) \otimes e = \\ &= (-d(a)b, 0) \otimes e. \end{aligned}$$

By definition

$$a(0, 1) = (d(a), a) = (d(a), 0) + (0, a) = (d(a), 0) + (0, 1)a.$$

It follows

$$(-d(a), 0) = (0, 1)a - a(0, 1).$$

We get

$$\begin{aligned} [d_E, a](be) &= (-d(a)b, 0) \otimes e = b(-d(a), 0) \otimes e = b((0, 1)a \otimes e - a(0, 1) \otimes e) = \\ &= b((0, 1) \otimes ae - a(0, 1) \otimes e) = b(d_E(ae) - ad_E(e)) = b[d_E, a](e). \end{aligned}$$

Hence $[d_E, a] \in \text{Hom}_B(E, \mathcal{J}_I^1(E)) = \text{Diff}^0(E, \mathcal{J}_I^1(E))$ for all $a \in B$. It follows $d_E \in \text{Diff}_A^1(E, \mathcal{J}_I^1(E))$ and the claim of the Lemma follows. \square

Recall there is a derivation $d_{\mathcal{J}} \in \text{Der}_A(\mathcal{J}_I^1, I)$ defined by $d_{\mathcal{J}}(x, a) = x + d(a)$. It follows $d_{\mathcal{J}}|_B = d_I$. Let F be a left \mathcal{J}_I^1 -module. An A -linear map

$$\nabla : F \rightarrow I \otimes_B F$$

satisfying

$$\nabla((x, a)f) = (x, a)\nabla(f) + d_{\mathcal{J}}(x, a) \otimes f$$

is an $(I, d_{\mathcal{J}})$ -connection on F . Assume E is a B -module. An A -linear map

$$\nabla : E \rightarrow I \otimes_B E$$

satisfying

$$\nabla(ae) = a\nabla(e) + d(a) \otimes e$$

is an (I, d) -connection.

Lemma 2.8. *Assume ∇ is an $(I, d_{\mathcal{J}})$ -connection. It follows ∇ is an (I, d) -connection.*

Proof. Let $b \in B$ and $f \in F$. We get

$$\begin{aligned} \nabla(bf) &= \nabla((0, b)f) = (0, b)\nabla(f) + d_{\mathcal{J}}(0, b) \otimes f = \\ &= b\nabla(f) + d(b) \otimes f \end{aligned}$$

and the Lemma is proved. \square

The sequence 2.3.1 is an exact sequence of left \mathcal{J}_I^1 -modules. We get an extension class

$$c_{\mathcal{J}}(E) \in \text{Ext}_{\mathcal{J}_I^1}^1(E, I \otimes_B E).$$

When we restrict to B we get an extension class

$$c_B(E) \in \text{Ext}_B^1(E, I \otimes_B E).$$

Definition 2.9. Let $c_{\mathcal{J}}(E)$ be the Atiyah class of E with respect to \mathcal{J}_I^1 . Let $c_B(E)$ be the Atiyah class of E with respect to B .

Proposition 2.10. *The following holds:*

- (2.10.1) *2.3.1 is split as sequence of right \mathcal{J}_I^1 -modules.*
- (2.10.2) *2.3.1 is left split as \mathcal{J}_I^1 -modules iff E has an $(I, d_{\mathcal{J}})$ -connection.*
- (2.10.3) *2.3.1 is left split as B -modules iff E has an (I, d) -connection.*

Proof. We prove claim 2.10.1: Define the following map:

$$s : E \rightarrow \mathcal{J}_I^1(E)$$

by

$$s(e) = (0, e).$$

It follows

$$s(e(x, a)) = s(ea) = (0, ea) = (0, e)(x, a) = s(e)(x, a)$$

and claim 2.10.1 is proved. We prove claim 2.10.2: Assume $s(e) = (\nabla(e), e)$ is a left \mathcal{J}_I^1 -linear section. We get

$$\begin{aligned} s((x, a)e) &= (\nabla((x, a)e), (x, a)e) = (\nabla((x, a)e), ae) = \\ (x, a)(\nabla(e), e) &= (x \otimes e + a\nabla(e) + d(a) \otimes e, ae) = ((x, a)\nabla(e) + (x + d(a)) \otimes e, ae). \end{aligned}$$

It follows ∇ satisfies

$$\nabla((x, a)e) = (x, a)\nabla(e) + d_{\mathcal{J}}(x, a) \otimes e$$

and ∇ is a $(I, d_{\mathcal{J}})$ -connection. Claim 2.10.2 follows. Claim 2.10.3 follows in a similar way and the Proposition is proved. \square

Example 2.11. *The classical Atiyah class.*

When $I = \Omega_{B/A} = \Omega$ and $d : B \rightarrow \Omega$ is the universal derivation it follows the class $c_B(E)$ is the classical Atiyah class as defined in [2]. From Proposition 2.10 it follows $c_B(E) = 0$ if and only if E has a connection

$$\nabla : E \rightarrow \Omega_{B/A} \otimes_B E$$

Corollary 2.12. *If $c_{\mathcal{J}}(E) = 0$ it follows $c_B(E) = 0$.*

Proof. If $c_{\mathcal{J}}(E) = 0$ it follows from Proposition 2.10, claim 2.10.2 E has an $(I, d_{\mathcal{J}})$ -connection. It follows from Lemma 5.2 E has a (I, d) -connection. From this and Proposition 2.10, claim 2.10.3 the Lemma follows. \square

Let $\mathcal{J}_I^1(E)^{left}$ denote the abelian group $\mathcal{J}_I^1(E)$ with its left B -module structure. Let $\mathcal{J}_I^1(E)^{right}$ denote the abelian group $\mathcal{J}_I^1(E)$ with its right B -module structure. We say the module I is *abelianized* if the following holds:

$$ax = xa$$

for all $a \in B$ and $x \in I$. Define the following product on I :

$$a * x = xa$$

for all $a \in B$ and $x \in I$. It follows we have defined an B -module structure on I , denoted I^{star} with the property there is an isomorphism

$$I^{right} \cong I^{star}$$

of A -modules. When we form the tensor product

$$I \otimes_B E$$

we use the right structure on I and left structure on E . Since B is commutative it follows E has a canonical right B -module structure. The abelian group $I \otimes_B E$ has a left and right structure as B -module, denoted

$$I \otimes_B E^{left}$$

and

$$I \otimes_B E^{right}.$$

We form a new product on $I \otimes_B E$ as follows:

$$a * (x \otimes e) = x \otimes (ea)$$

for any $a \in B$ and $x \otimes e \in I \otimes_B E$. We get a left B -module denoted $I \otimes_B E^{star}$. There is an isomorphism

$$I \otimes_B E^{star} \cong I \otimes_B E^{right}$$

of B -modules.

Lemma 2.13. *Assume I is abelianized. It follows there is an isomorphism*

$$I \otimes_B E^{left} \cong I \otimes_B E^{right}$$

of B -modules.

Proof. Define the following map:

$$\phi : I \otimes_B E^{left} \rightarrow I \otimes_B E^{star}$$

by

$$\phi(x \otimes e) = x \otimes e.$$

We get

$$\begin{aligned} \phi(a(x \otimes e)) &= \phi((ax) \otimes e) = (ax) \otimes e = (xa) \otimes e = \\ &= x \otimes (ae) = x \otimes (ea) = a * (x \otimes e) = a * \phi(x \otimes e). \end{aligned}$$

It follows

$$I \otimes_B E^{left} \cong I \otimes_B E^{star} \cong I \otimes_B E^{right}$$

and the Lemma follows. \square

Proposition 2.14. *Assume I is abelianized. The following holds:*

$$(2.14.1) \quad c_B(E) = 0 \text{ iff } \mathcal{J}_I^1(E)^{left} \cong \mathcal{J}_I^1(E)^{right} \text{ as } B\text{-modules.}$$

$$(2.14.2) \quad \mathcal{J}_I^1(E)^{left} \cong \mathcal{J}_I^1(E)^{right} \text{ as abelian groups.}$$

$$(2.14.3) \quad \mathcal{J}_I^1(E)^{left} \cong \mathcal{J}_I^1(E)^{right} \text{ as } B\text{-modules iff } E \text{ has an } (I, d)\text{-connection}$$

Proof. It follows $c_B(E) = 0$ if and only if the sequence

$$0 \rightarrow I \otimes_B E \rightarrow \mathcal{J}_I^1(E) \rightarrow E \rightarrow 0$$

is split as sequence of left B -modules. From Lemma 2.13 it follows

$$\begin{aligned} \mathcal{J}_I^1(E)^{left} &\cong I \otimes_B E \oplus E^{left} \cong \\ &I \otimes_A E \oplus E^{right} \cong \mathcal{J}_I^1(E)^{right} \end{aligned}$$

since by Proposition 2.10, claim 2.10.1 the sequence is always split as right B -modules. Claim 2.14.1 follows. Claim 2.14.2 is obvious. Claim 2.14.3: From 2.14.1 we get

$$\mathcal{J}_I^1(E)^{left} \cong \mathcal{J}_I^1(E)^{right}$$

as B -modules iff $c_B(E) = 0$. From Proposition 2.10, claim 2.10.3 it follows $c_B(E) = 0$ iff E has an (I, d) -connection. Claim 2.14.3 is proved and the Proposition follows. \square

Hence the characteristic class $c_B(E)$ measures when the abelian group $\mathcal{J}_I^1(E)$ may be equipped with two non isomorphic structures as B -module.

3. GENERALIZED ATIYAH CLASSES FOR QUASI COHERENT SHEAVES

In this section we generalize the results in the previous section to the case where we consider an arbitrary morphism $\pi : X \rightarrow S$ of schemes and an arbitrary quasi coherent \mathcal{O}_X -module \mathcal{E} . We define the generalized Atiyah class $c_{\mathcal{J}}(\mathcal{E})$ and $c_{\mathcal{O}_X}(\mathcal{E})$ using derivations and sheaves of associative rings and prove various properties of this construction. We end the section with a discussion of explicit examples. We give examples of jet bundles $\mathcal{J}_{\Omega}^1(\mathcal{O}(d))$ where $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$ and $\mathcal{O}(1)$ is the tautological quotient bundle on \mathbf{P}_K^1 . We prove $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d)) \neq 0$ when $\text{char}(K) = 0$ and $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d)) = 0$ when $\text{char}(K)$ divides d . Hence the characteristic classes $c_{\mathcal{J}}$ and $c_{\mathcal{O}_X}$ are non trivial.

Let in the following $\pi : X \rightarrow S$ be an arbitrary morphism of schemes and let \mathcal{I} be an arbitrary quasi coherent left and right \mathcal{O}_X -module. Let

$$d \in \text{Der}_{\pi^{-1}(\mathcal{O}_S)}(\mathcal{O}_X, \mathcal{I})$$

be a derivation. We may form the quasi coherent sheaf $\mathcal{J}_{\mathcal{I}}^1 = \mathcal{I} \oplus \mathcal{O}_X$. Let $V \subseteq U$ be open subsets of X and let $(x, a), (y, b) \in \mathcal{J}_{\mathcal{I}}^1(U)$ be two elements. Define

$$(x, a)(y, b) = (xb + ay, ab).$$

It follows $\mathcal{J}_{\mathcal{I}}^1(U)$ is an associative ring with multiplicative unit $\mathbf{1} = (0, 1)$ and the restriction morphism

$$\mathcal{J}_{\mathcal{I}}^1(U) \rightarrow \mathcal{J}_{\mathcal{I}}^1(V)$$

is a morphism of unital rings. There is a natural embedding

$$i : \mathcal{I} \rightarrow \mathcal{J}_{\mathcal{I}}^1$$

defined as follows:

$$i(U) : \mathcal{I}(U) \rightarrow \mathcal{J}_{\mathcal{I}}^1(U)$$

$$i(U)(x) = (x, 0).$$

It follows $\mathcal{I} \subseteq \mathcal{J}_{\mathcal{I}}^1$ is a sheaf of ideals in $\mathcal{J}_{\mathcal{I}}^1$ with $\mathcal{I}^2 = 0$. We get an exact sequence

$$(3.0.4) \quad 0 \rightarrow \mathcal{I} \rightarrow \mathcal{J}_{\mathcal{I}}^1 \rightarrow \mathcal{O}_X \rightarrow 0$$

of sheaves of abelian groups. The sequence 3.0.4 is an extension of \mathcal{O}_X by a quasi coherent sheaf of two sided ideals of square zero.

We may form the quasi coherent sheaf $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}$. We may define a left and right \mathcal{O}_X -structure and a left $\mathcal{J}_{\mathcal{I}}^1$ -module structure on $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ as follows: Let $(x, a) \in \mathcal{J}_{\mathcal{I}}^1(U)$ and $(z \otimes e, f) \in \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(U)$. Let

$$(x, a)(z \otimes e, f) = (x \otimes f + az \otimes e + d(a) \otimes f, af).$$

and

$$(z \otimes e, f)(x, a) = (z \otimes ea, fa).$$

One checks for any open sets $V \subseteq U$ it follows the restriction morphism

$$\rho_{UV} : \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(U) \rightarrow \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(V)$$

satisfies

$$(x, a)(z \otimes e, f)|_V = (x, a)|_V(z \otimes e, f)|_V.$$

For open sets $W \subseteq V \subseteq U$ it follows

$$\rho_{VW} \circ \rho_{UV} = \rho_{UW}.$$

Similar formulas holds for the right structure as \mathcal{J}_T^1 -module. It follows $\mathcal{J}_T^1(\mathcal{E})$ becomes a sheaf of left and right \mathcal{J}_T^1 -modules and left and right \mathcal{O}_X -modules. We get an exact sequence

$$(3.0.5) \quad 0 \rightarrow \mathcal{I} \otimes_{\mathcal{O}} \mathcal{E} \rightarrow \mathcal{J}_T^1(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow 0$$

of sheaves left and right \mathcal{O}_X -modules and left and right \mathcal{J}_T^1 -modules. These assertions follow immediately from the local situation since all sheaves involved are quasi coherent.

Definition 3.1. Let the sequence 3.0.5 be the *Atiyah-Karoubi sequence of \mathcal{E} with respect to (\mathcal{I}, d)* .

Define the following $\pi^{-1}(\mathcal{O}_S)$ -linear map:

$$d_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{J}_T^1(\mathcal{E})$$

by

$$d_{\mathcal{E}}(U)(e) = (0, 1) \otimes e \in \mathcal{J}_T^1(\mathcal{E})(U).$$

Here $U \subseteq X$ is an open subset.

Lemma 3.2. *The following holds:*

$$d_{\mathcal{E}} \in \text{Diff}_{\pi^{-1}(\mathcal{O}_S)}^1(\mathcal{E}, \mathcal{J}_T^1(\mathcal{E})).$$

Proof. The Lemma follows from Lemma 2.7 since the sheaf \mathcal{E} is quasi coherent. \square

The morphism $d_{\mathcal{E}}$ is the *universal differential operator for \mathcal{E} with respect to (\mathcal{I}, d)* .

Example 3.3. *The first order jet bundle.*

Assume $\mathcal{I} = \Omega_{X/S}^1$ and d the universal derivation. We get a map

$$d_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{J}_{X/S}^1(\mathcal{E})$$

defined by

$$d_{\mathcal{E}}(U)(e) = 1 \otimes e \in \mathcal{J}_{X/S}^1(\mathcal{E})(U).$$

This map is the classical differential operator $d_{\mathcal{E}} \in \text{Diff}_{\pi^{-1}(\mathcal{O}_S)}^1(\mathcal{E}, \mathcal{J}_{X/S}^1(\mathcal{E}))$ for \mathcal{E} .

In the following we view the sequence 3.0.5 as an exact sequence of sheaves of left \mathcal{J}_T^1 and \mathcal{O}_X -modules. We get a characteristic class

$$c_{\mathcal{J}}(\mathcal{E}) \in \text{Ext}_{\mathcal{J}_T^1}^1(\mathcal{E}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E})$$

and

$$c_{\mathcal{O}_X}(\mathcal{E}) \in \text{Ext}_{\mathcal{O}_X}^1(\mathcal{E}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E})$$

Definition 3.4. Let $c_{\mathcal{J}}(\mathcal{E})$ be the *Atiyah class of \mathcal{E} with respect to \mathcal{J}_T^1* . Let $c_{\mathcal{O}_X}(\mathcal{E})$ be the *Atiyah class of \mathcal{E} with respect to \mathcal{O}_X* .

There is a derivation $d_{\mathcal{J}} \in \text{Der}_{\pi^{-1}(\mathcal{O}_S)}(\mathcal{J}_{\mathcal{I}}^1, \mathcal{I})$ defined as follows: Let $(x, a) \in \mathcal{J}_{\mathcal{I}}^1(U)$. Define

$$d_{\mathcal{J}}(x, a) = x + d(a).$$

Assume \mathcal{E} is a quasi coherent \mathcal{O}_X -module. We say an $\pi^{-1}(\mathcal{O}_S)$ -linear map

$$\nabla : \mathcal{E} \rightarrow \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E}$$

is an (\mathcal{I}, d) -connection if for all local sections $a \in \mathcal{O}(U)$ and $e \in \mathcal{E}(U)$ on an open set $U \subseteq X$ the following holds:

$$\nabla(ae) = a\nabla(e) + d(a) \otimes e.$$

Assume \mathcal{F} is a left $\mathcal{J}_{\mathcal{I}}^1$ -module which is quasi coherent as left \mathcal{O}_X -module. We say an $\pi^{-1}(\mathcal{O}_S)$ -linear map

$$\nabla : \mathcal{F} \rightarrow \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an $(\mathcal{I}, d_{\mathcal{J}})$ -connection if the following holds for $(x, a) \in \mathcal{J}_{\mathcal{I}}^1(U)$ and $f \in \mathcal{F}(U)$:

$$\nabla((x, a)f) = (x, a)\nabla(f) + d_{\mathcal{J}}(x, a) \otimes f.$$

Proposition 3.5. *The following holds:*

$$(3.5.1) \quad c_{\mathcal{J}}(\mathcal{E}) = 0 \text{ iff } \mathcal{E} \text{ has an } (\mathcal{I}, d_{\mathcal{J}})\text{-connection.}$$

$$(3.5.2) \quad c_{\mathcal{O}_X}(\mathcal{E}) = 0 \text{ iff } \mathcal{E} \text{ has an } (\mathcal{I}, d)\text{-connection.}$$

$$(3.5.3) \quad \text{If } c_{\mathcal{J}}(\mathcal{E}) = 0 \text{ it follows } c_{\mathcal{O}_X}(\mathcal{E}) = 0.$$

Proof. The proof is left to the reader. \square

We say the sheaf \mathcal{I} is *abelianized* if for all local sections a of \mathcal{O}_X and x of \mathcal{I} the following holds

$$ax = xa.$$

As in Lemma 2.13 We get an isomorphism

$$I \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}^{\text{left}} \cong I \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}^{\text{right}}$$

of \mathcal{O}_X -modules.

Theorem 3.6. *Assume \mathcal{I} is abelianized. The following holds:*

$$(3.6.1) \quad \text{The sequence 3.0.5 is split as sequence of right } \mathcal{O}_X\text{-modules}$$

$$(3.6.2) \quad \text{3.0.5 is split as left } \mathcal{J}_{\mathcal{I}}^1\text{-module iff } \mathcal{E} \text{ has an } (\mathcal{I}, d_{\mathcal{J}})\text{-connection.}$$

$$(3.6.3) \quad \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{\text{left}} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{\text{right}} \text{ as sheaves of abelian groups.}$$

$$(3.6.4) \quad c_{\mathcal{O}_X}(\mathcal{E}) = 0 \text{ iff } \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{\text{left}} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{\text{right}} \text{ as } \mathcal{O}_X\text{-modules.}$$

$$(3.6.5) \quad \text{3.0.5 is split as left } \mathcal{O}_X\text{-modules iff } \mathcal{E} \text{ has an } (\mathcal{I}, d)\text{-connection}$$

Proof. We prove 3.6.1: Let $U \subseteq X$ be an open subset and define the morphism

$$s(U) : \mathcal{E}(U) \rightarrow \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(U)$$

by

$$s(U)(e) = (0, e).$$

Let $X = (x \otimes a, b) \in \mathcal{J}_{\mathcal{I}}^1(U)$. It follows

$$s(U)(eX) = (0, eb) = (0, e)X = s(U)(e)X$$

hence s is left $\mathcal{J}_{\mathcal{I}}^1$ -linear. The map s splits 3.0.5 as sequence of left $\mathcal{J}_{\mathcal{I}}^1$ -modules and claim 3.6.1 follows. We prove 3.6.2: By definition 3.0.5 is split as left $\mathcal{J}_{\mathcal{I}}^1$ -modules iff $c_{\mathcal{J}}(\mathcal{E}) = 0$. By proposition this is iff \mathcal{E} has an $(\mathcal{I}, d_{\mathcal{J}})$ -connection.

Claim 3.6.2 is proved. We prove 3.6.5: The sequence 3.0.5 is split as left \mathcal{O}_X -modules iff $c_{\mathcal{O}_X}(\mathcal{E}) = 0$. By Proposition 3.5 this is iff \mathcal{E} has an (\mathcal{I}, d) -connection. Claim 3.6.5 is proved. We prove 3.6.4: By definition it follows

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{right} \cong \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}^{right}$$

as right \mathcal{O}_X -modules. Sequence 3.0.5 is left split as \mathcal{O}_X -modules iff

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{left} \cong \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}^{left}.$$

It follows $c_{\mathcal{O}_X}(\mathcal{E}) = 0$ iff there is an isomorphism

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{left} \cong \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}^{left} \cong \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}^{right} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{right}.$$

Claim 3.6.4 follows. Claim 3.6.3 is obvious and the Theorem follows. \square

Hence the characteristic class $c_{\mathcal{O}_X}(\mathcal{E})$ measures when the sheaf of abelian groups $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ is equipped with two non isomorphic structures as \mathcal{O}_X -module.

Example 3.7. *The classical case: $\mathcal{I} = \Omega_{X/S}^1$.*

Assume in the following Proposition $\pi : X \rightarrow S$ is a separated morphism. Let $\Delta : X \rightarrow X \times_S X$ be the diagonal embedding. It follows $\Delta(X) \subseteq X \times_S X$ is a closed subscheme. Let $\mathcal{J} \subseteq \mathcal{O}_{X \times_S X}$ be the ideal sheaf of $\Delta(X)$ and let $\mathcal{O}_{\Delta^l} = \mathcal{O}_{X \times_S X} / \mathcal{J}^{l+1}$ be the l 'th infinitesimal neighborhood of the diagonal. Let $p, q : X \times_S X \rightarrow X$ be the canonical projection maps.

Definition 3.8. Let $\mathcal{J}_{X/S}^l(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^{l+1}} \otimes q^* \mathcal{E})$ be the l 'th order jet bundle of \mathcal{E} .

Assume π is given by a homomorphism $\phi : A \rightarrow B$ of commutative rings. Let $X = \text{Spec}(B)$ and $S = \text{Spec}(A)$. Assume \mathcal{E} is the sheafification of a B -module E . It follows $\mathcal{J}_{X/S}^1(\mathcal{E})$ is the sheaf associated to $P_{B/A}^1(E) = B \otimes_A B / J^{l+1} \otimes_B E$ where $J \subseteq B \otimes_A B$ is the kernel of the multiplication map.

Proposition 3.9. *Assume $\mathcal{I} = \Omega_{X/S}$ and $d : \mathcal{O}_X \rightarrow \Omega_{X/S}$ is the universal derivation. It follows $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \cong \mathcal{J}_{X/S}^1(\mathcal{E})$ is the first order jet bundle of \mathcal{E} . The generalized Atiyah sequence 3.0.5 becomes the classical Atiyah sequence.*

Proof. Assume $U = \text{Spec}(B) \subseteq X$ is an open affine subscheme mapping to an open affine subscheme $V = \text{Spec}(A) \subseteq S$. Let $\mathcal{E}(U) = E$ where E is a B -module and let $\Omega_{X/S}|(U) = \Omega$. Let $m : B \otimes_A B \rightarrow B$ be the multiplication map and let $s : B \rightarrow P_{B/A}^1$ be defined by $s(b) = 1 \otimes b$. It follows there is an isomorphism

$$\phi : P_{B/A}^1 \otimes_B E \rightarrow \Omega \otimes_B E \oplus E$$

defined by

$$\phi(x \otimes e) = ((x - sm(x)) \otimes e, m(x)e).$$

One checks we get a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega \otimes_B E & \longrightarrow & P_{B/A}^1(E) & \longrightarrow & E \longrightarrow 0 \\ & & \downarrow = & & \downarrow \cong & & \downarrow = \\ 0 & \longrightarrow & \Omega \otimes_B E & \longrightarrow & \mathcal{J}_{\Omega}^1(E) & \longrightarrow & E \longrightarrow 0 \end{array}$$

where the middle vertical arrow is the isomorphism ϕ . Since the map ϕ is intrinsically defined it glues to give an isomorphism $\mathcal{J}_{X/S}^1(\mathcal{E}) \cong \mathcal{J}_{\Omega}^1(\mathcal{E})$. The Proposition follows. \square

Note: If $I = \Omega_{B/A}$ and $d : B \rightarrow \Omega_{B/A}$ is the universal derivation the construction of $\mathcal{J}_{\Omega_{B/A}}^1(E)$ using d is due to Karoubi (see [6]). In this case we get the sequence

$$0 \rightarrow \Omega_{B/A} \otimes_B E \rightarrow P_{B/A}^1(E) \rightarrow E \rightarrow 0$$

where $P_{B/A}^1(E)$ is the *first order jet module* of E . It is also called the *first order module of principal parts* of E .

Note: Assume \mathcal{E} is a finite rank locally free \mathcal{O}_X -module. If X is a smooth scheme of finite type over \mathbf{C} - the complex numbers, $\mathcal{I} = \Omega_X^1$ is the module of differentials and d the universal derivation we get the classical Atiyah sequence

$$0 \rightarrow \Omega_X^1 \otimes \mathcal{E} \rightarrow \mathcal{J}_{\Omega_X^1}^1(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow 0.$$

The class

$$c_{\mathcal{O}_X}(\mathcal{E}) \in \text{Ext}_{\mathcal{O}_X}^1(\mathcal{E}, \Omega_X^1 \otimes \mathcal{E})$$

is the classical Atiyah class as defined in [2]. It follows $c_{\mathcal{O}_X}(\mathcal{E}) = 0$ iff \mathcal{E} has a connection

$$\nabla : \mathcal{E} \rightarrow \Omega_X^1 \otimes \mathcal{E}.$$

Example 3.10. *Sheaves of jets on the projective line.*

Let K be a field of characteristic zero and consider \mathbf{P}_K^1 . Let $\mathcal{O}(d)$ be $\mathcal{O}(1)^{\otimes d}$ where $\mathcal{O}(1)$ is the tautological quotient bundle on \mathbf{P}_K^1 . Let $\Omega = \Omega_{\mathbf{P}_K^1}^1$ be the sheaf of differentials on \mathbf{P}_K^1 . We get from 3.9 an exact sequence of $\mathcal{O}_{\mathbf{P}^1}$ -modules

$$0 \rightarrow \Omega \otimes \mathcal{O}(d) \rightarrow \mathcal{J}_{\Omega}^1(\mathcal{O}(d)) \rightarrow \mathcal{O}(d) \rightarrow 0.$$

It follows $\mathcal{J}_{\Omega}^1(\mathcal{O}(d)) \cong \mathcal{J}_{\mathbf{P}^1}^1(\mathcal{O}(d))$ is the first order jet bundle of $\mathcal{O}(d)$ on \mathbf{P}_K^1 . There is a left and right $\mathcal{O}_{\mathbf{P}^1}$ -module structure on $\mathcal{J}_{\mathbf{P}_K^1}^1(\mathcal{O}(d))$ and by [10] isomorphisms

$$\mathcal{J}_{\mathbf{P}_K^1}^1(\mathcal{O}(d))^{\text{left}} \cong \mathcal{O}(d-1) \oplus \mathcal{O}(d-1)$$

and

$$\mathcal{J}_{\mathbf{P}_K^1}^1(\mathcal{O}(d))^{\text{right}} \cong \mathcal{O}(d) \oplus \mathcal{O}(d-2)$$

of $\mathcal{O}_{\mathbf{P}^1}$ -modules. It follows from Theorem 3.6, claim 3.6.4 $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d)) \neq 0$. There is an isomorphism

$$\mathcal{J}_{\mathbf{P}_K^1}^1(\mathcal{O}(d))^{\text{left}} \cong \mathcal{J}_{\mathbf{P}_K^1}^1(\mathcal{O}(d))^{\text{right}}$$

as sheaves of abelian groups.

By Proposition 3.5 it follows $c_{\mathcal{J}}(\mathcal{O}(d)) \neq 0$ if $\text{char}(K) = 0$ hence the classes $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d))$ and $c_{\mathcal{J}}(\mathcal{O}(d))$ are non trivial.

Again by [10] if $\text{char}(K)$ divides d it follows there is an isomorphism

$$\mathcal{J}_{\mathbf{P}_K^1}^1(\mathcal{O}(d))^{\text{left}} \cong \mathcal{J}_{\mathbf{P}_K^1}^1(\mathcal{O}(d))^{\text{right}}$$

of $\mathcal{O}_{\mathbf{P}^1}$ -modules. Hence in this case $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d)) = 0$. The sheaf $\mathcal{J}_{\mathbf{P}^1}^1(\mathcal{O}(d))$ and the class $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d))$ is defined over $\mathbf{P}_{\mathbf{Z}}^1$. When we pass to \mathbf{Q} the class $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d))$ is non zero. When we pass to $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and when p divides d the class $c_{\mathcal{O}_{\mathbf{P}^1}}(\mathcal{O}(d))$ becomes zero.

4. JETS AND INFINITESIMAL EXTENSIONS OF SHEAVES

In this section we define and study jet bundles $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ of \mathcal{O}_X -modules \mathcal{E} relative to any morphism $\pi : X \rightarrow S$ of schemes. We define in Definition 4.2 the generalized Atiyah sequence of \mathcal{E} with respect to a square zero extension \mathcal{J} and a derivation $d \in \text{Der}_{\mathcal{R}}(\mathcal{O}_X, \mathcal{I})$. Here \mathcal{J} is any sheaf of associative unital algebras on X that are square zero extensions of \mathcal{O}_X with respect to a quasi coherent left and right \mathcal{O}_X -module \mathcal{I} . We use the generalized Atiyah sequence 4.2 in Definition 4.4 to define the generalized Atiyah classes $c_{\mathcal{J}}(\mathcal{E})$ and $c_{\mathcal{O}_X}(\mathcal{E})$ for \mathcal{E} . This construction generalize the classical construction (see Example 4.5).

Let in the following $\pi : X \rightarrow S$ be a morphism of schemes and let $\mathcal{R} = \pi^{-1}(\mathcal{O}_S)$. It follows \mathcal{O}_X is a sheaf of \mathcal{R} -algebras. This means for any open subset $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is an $\mathcal{R}(U)$ -algebra. Let \mathcal{E} be a quasi coherent sheaf of left \mathcal{O}_X -modules. It follows \mathcal{E} is canonically a sheaf of quasi coherent right \mathcal{O}_X -modules. Let

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{J} \xrightarrow{m} \mathcal{O}_X \rightarrow 0$$

be an infinitesimal extension of \mathcal{O}_X by a quasi coherent left and right \mathcal{O}_X -module \mathcal{I} . This means \mathcal{J} is a sheaf of associative unital rings on X and $\mathcal{I} \subseteq \mathcal{J}$ is a sheaf of quasi coherent two sided ideals with $\mathcal{I}^2 = 0$. Let $d \in \text{Der}_{\mathcal{R}}(\mathcal{O}_X, \mathcal{I})$ be a derivation over \mathcal{R} . This means for any open set U $d(U) \in \text{Der}_{\mathcal{R}(U)}(\mathcal{O}_X(U), \mathcal{I}(U))$ is a derivation over $\mathcal{R}(U)$. The derivation d induce a derivation $p = d \circ m$ in $\text{Der}_{\mathcal{R}}(\mathcal{J}, \mathcal{I})$. Make the following definition

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \oplus \mathcal{E}.$$

Let $a \in \mathcal{J}(U)$ and $Z = (z \otimes e, f) \in \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(U)$ and define

$$aZ = a(z \otimes e, f) = ((az) \otimes e + p(a) \otimes f, af)$$

and

$$Za = (z \otimes e, f)a = (z \otimes (ea), fa).$$

this is well defined since the sheaf \mathcal{O}_X is a sheaf of commutative rings hence \mathcal{E} is canonically a sheaf of right \mathcal{O}_X -modules. It follows for any open subset $V \subseteq U$ there is an equality

$$a(z \otimes e, f)|_V = a|_V(z \otimes e, f)|_V.$$

If ρ_{UV} is the restriction map from

$$\rho_{UV} : \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(U) \rightarrow \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(V)$$

it follows for any open sets $W \subseteq V \subseteq U$ we get

$$\rho_{VW} \circ \rho_{UV} = \rho_{UW}.$$

Let $W = (w \otimes g, h) \in \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(U)$ and $b \in \mathcal{J}(U)$.

Proposition 4.1. *The sheaf $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ is a sheaf of left and right \mathcal{J} -modules. The sheaf $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E}$ is a sheaf of left and right \mathcal{J} -submodules of $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$. The sheaf \mathcal{E} is a sheaf of left and right \mathcal{J} -modules.*

Proof. The proof is left to the reader. \square

Note: $\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})$ is not a lifting of \mathcal{E} to \mathcal{J} in the sense of deformation theory since $\mathcal{I}\mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) = 0$. It follows

$$\mathcal{O}_X \otimes_{\mathcal{J}} \mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \cong (\mathcal{J}/\mathcal{I}) \otimes_{\mathcal{J}} \mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \cong$$

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})/\mathcal{I}\mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \neq \mathcal{E}.$$

By definition a lifting $\mathcal{E}_{\mathcal{J}}$ of \mathcal{E} to \mathcal{J} is required to satisfy

$$\mathcal{O}_X \otimes_{\mathcal{J}} \mathcal{E}_{\mathcal{J}} \cong \mathcal{E}.$$

We get a natural exact sequence of sheaves of abelian groups

$$(4.1.1) \quad 0 \rightarrow \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow^i \mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \rightarrow^j \mathcal{E} \rightarrow 0.$$

Definition 4.2. Let the sequence 4.1.1 be the *Atiyah-Karoubi sequence* of \mathcal{E} with respect to the pair (\mathcal{I}, d) .

Corollary 4.3. *The sequence 4.1.1 is an exact sequence of left and right \mathcal{J} -modules and left and right \mathcal{O}_X -modules.*

Proof. One checks the natural maps i, j are left and right \mathcal{J} -linear, left and right \mathcal{O}_X -linear and the Corollary follows. \square

View the sequence 4.1.1 as a sequence of left \mathcal{J} and \mathcal{O}_X -modules. We get two characteristic classes:

$$c_{\mathcal{J}}(\mathcal{E}) \in \text{Ext}_{\mathcal{J}}^1(\mathcal{E}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E})$$

and

$$c_{\mathcal{O}_X}(\mathcal{E}) \in \text{Ext}_{\mathcal{O}_X}^1(\mathcal{E}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Definition 4.4. The class $c_{\mathcal{J}}(\mathcal{E})$ is the *Atiyah class* of \mathcal{E} with respect to \mathcal{J} . The class $c_{\mathcal{O}_X}(\mathcal{E})$ is the *Atiyah class* of \mathcal{E} with respect to \mathcal{O}_X .

Example 4.5. *The classical Atiyah class.*

The class $c_{\mathcal{O}_X}(\mathcal{E})$ generalize the classical Atiyah class: When X is a separated scheme over a fixed base scheme S and $\mathcal{I} = \Omega_{X/S}^1$ is the sheaf of differentials, d is the universal derivation, $\mathcal{J} = \mathcal{I} \times \mathcal{O}_X$ is the trivial square zero extension and $\mathcal{R} = \pi^{-1}(\mathcal{O}_S)$ it follows $c_{\mathcal{O}_X}(\mathcal{E})$ is the classical Atiyah class as defined in [2].

5. APPENDIX: EXPLICIT EXAMPLES

In this section we give some explicit examples to illustrate the constructions made in the previous sections. Let in the following K be an arbitrary field.

Example 5.1. *Connections on projective modules.*

Let A be a commutative ring with unit and let P be a finitely generated projective A -module. Let

$$0 \rightarrow K \rightarrow A^n \xrightarrow{p} P \rightarrow 0$$

be an exact sequence where A^n is the free A -module of rank n . Since P is projective the map q has a section s with $q \circ s = \text{id}$.

Let $F = A^n = A\{e_1, \dots, e_n\}$. Let $p_i : F \rightarrow P$ be defined by $p_i = e_i^*$. Let $q_i = p_i \circ s$. We get elements

$$e_1, \dots, e_n \in P$$

and

$$q_1, \dots, q_n \in P^*$$

satisfying the following:

$$\sum_{i=1}^n q_i(a) e_i = a$$

for any $a \in P$. Define the following map:

$$\nabla(e) = \sum_{i=1}^n d(q_i(e)) \otimes e_i.$$

Lemma 5.2. *The map ∇ is a connection on P .*

Proof. The map ∇ is by definition well-defined. We check it is a connection.

$$\begin{aligned} \nabla(ae) &= \sum_i d(q_i(ae)) \otimes e_i = \sum_i d(aq_i(e)) \otimes e_i = \\ &= \sum_i (ad(q_i(e)) + d(a)q_i(e)) \otimes e_i = a \sum_i d(q_i(e)) \otimes e_i + d(a) \otimes \sum_i q_i(e)e_i = \\ &= a\nabla(e) + d(a) \otimes e \end{aligned}$$

and the map ∇ is a connection. The Lemma follows. \square

The universal derivation $d : A \rightarrow \Omega^1$ gives rise to the Atiyah-Kaorubi sequence

$$0 \rightarrow \Omega^1 \otimes P \rightarrow \mathcal{J}_{\Omega^1}^1(P) \rightarrow P \rightarrow 0.$$

The abelian group $\mathcal{J}^1(P) = \mathcal{J}_{\Omega^1}^1(P)$ has a canonical left A -structure and a canonical right A -structure. It has by the previous section a non-trivial left A -structure defined as follows:

$$a(x \otimes e, f) = ((ax) \otimes e + d(a) \otimes f, af)$$

with $a \in A$ and $(x \otimes e, f) \in \mathcal{J}^1(P)$. By Lemma 5.2 and Theorem 3.6 there is an isomorphism

$$\mathcal{J}^1(P)^{left} \cong \mathcal{J}^1(P)^{right}$$

of A -modules.

Example 5.3. *Generalized Atiyah classes on the projective line*

In this example we consider generalized Atiyah-sequences of locally free sheaves on the projective line. We prove that in the case of an invertible sheaf the only non-trivial Atiyah-class arise in the classical case. The aim of the constructions made is to construct new examples of locally free sheaves of left and right modules on the projective line where the splitting type as left module differs from the splitting type as right module.

Let $V = K\{e_0, e_1\}$ and $V^* = K\{x_0, x_1\}$ and $\mathbf{P} = \mathbf{P}(V^*)$ the projective line over K . Let $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ be the m 'th tensor power of the tautological line bundle on \mathbf{P} . There is an isomorphism

$$\Omega_{\mathbf{P}}^1 \cong \mathcal{O}(-2)$$

where $\Omega_{\mathbf{P}}^1$ is the cotangent bundle on \mathbf{P} . We get an exact sequence

$$0 \rightarrow \Omega_{\mathbf{P}}^1 \otimes \mathcal{O}(m) \rightarrow \mathcal{J}_{\mathbf{P}}(\mathcal{O}(m)) \rightarrow \mathcal{O}(m) \rightarrow 0$$

called the *classical Atiyah sequence* of $\mathcal{O}(m)$. Assume

$$d \in \text{Der}_K(\mathcal{O}_{\mathbf{P}}, \mathcal{O}(m))$$

is a derivation and consider the generalized Atiyah-sequence

$$(5.3.1) \quad 0 \rightarrow \mathcal{O}(m) \otimes \mathcal{O}(l) \rightarrow \mathcal{J}_{\mathcal{O}(m)}(\mathcal{O}(l)) \rightarrow \mathcal{O}(l) \rightarrow 0$$

with respect to $(\mathcal{O}(m), d)$. We get a characteristic class

$$c_{\mathcal{O}_{\mathbf{P}}}(\mathcal{O}(l)) \in \text{Ext}^1(\mathcal{O}(l), \mathcal{O}(m+l)).$$

Proposition 5.4. *The only non-trivial case is when $m = -2$,*

$$d : \mathcal{O}_{\mathbf{P}} \rightarrow \Omega_{\mathbf{P}}^1$$

is the universal derivation and 5.3.1 is the classical Atiyah sequence.

Proof. The proof is left to the reader. \square

When we use a derivation $d : \mathcal{O}_{\mathbf{P}} \rightarrow \mathcal{I}$ where \mathcal{I} is a higher rank locally free sheaf we may get new examples.

Let $\mathcal{I} = \mathcal{J}_{\mathbf{P}} \cong \mathcal{O}(-2) \oplus \mathcal{O}$ be the first order jet bundle on \mathbf{P} . Define the following derivation $d : \mathcal{O}_{\mathbf{P}} \rightarrow \mathcal{I}$. Let $U_i = D(x_i)$ and let $t = x_1/x_0$, $s = 1/t$ be local coordinates on \mathbf{P} . Let $\mathcal{O}(U_0) = K[t]$, $\mathcal{I}(U_0) = K[t]\{dt, e\}$ and

$$d_0 : K[t] \rightarrow K[t]\{dt, e\}$$

be defined by

$$d_0(a(t)) = a'(t)dt + t^i a'(t)e$$

with $i = 0, 1, 2$. Let $\mathcal{O}(U_1) = K[s]$, $\mathcal{I}(U_1) = K[s]\{ds, f\}$ and

$$d_1 : K[s] \rightarrow K[s]\{ds, f\}$$

be defined by

$$d_1(b(s)) = b'(s)ds - s^{2-i}b'(s)f.$$

One checks d_0, d_1 glue to a derivation $d \in \text{Der}(\mathcal{O}_{\mathbf{P}}, \mathcal{I})$. We get for any linebundle $\mathcal{O}(l)$ on \mathbf{P} an Atiyah-Karoubi sequence

$$0 \rightarrow I \otimes \mathcal{O}(l) \rightarrow \mathcal{J}_{\mathcal{I}}(\mathcal{O}(l)) \rightarrow \mathcal{O}(l) \rightarrow 0.$$

The local structure of $\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))$ looks as follows:

$$\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))(U_0) = K[t]\{dt \otimes x_0^l, e \otimes x_0^l\}$$

with left $K[t]$ -multiplication given as follows:

$$a(\omega, bx_0^l) = (a\omega + d_0(a) \otimes bx_0^l, abx_0^l).$$

Moreover

$$\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))(U_1) = K[s]\{ds \otimes x_1^l, f \otimes x_1^l\}$$

with left $K[s]$ -multiplication given as follows:

$$c(\eta, dx_1^l) = (c\eta + d_1(c) \otimes dx_1^l, cdx_1^l).$$

We aim to construct the structure matrix of the locally free sheaf $\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))$ as left $\mathcal{O}_{\mathbf{P}}$ -module and to see how this matrix depend on the derivation d and the integer l . The splitting type of $\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))$ as right $\mathcal{O}_{\mathbf{P}}$ -module is by the previous section as follows:

$$\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))^{\text{right}} \cong \mathcal{O}(l-2) \oplus \mathcal{O}(l) \oplus \mathcal{O}(l).$$

Using Atiyah-Karoubi sequences we hope to give new examples where the splitting type as left module differs from the one as right module. Let $\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))(U_0)$ have the following basis as left $K[t]$ -module:

$$B : (1, 0, 0)_0 = dt \otimes x_0^l, (0, 1, 0)_0 = e \otimes x_0^l, (0, 0, 1)_0 = x_0^l.$$

Let $\mathcal{J}_{\mathcal{I}}(\mathcal{O}(l))(U_1)$ have the following basis as left $K[s]$ -module:

$$C : (1, 0, 0)_1 = ds \otimes x_1^l, (0, 1, 0)_1 = f \otimes x_1^l, (0, 0, 1)_1 = x_1^l.$$

Theorem 5.5. *The structure matrix $[L]_B^C$ is as follows:*

$$[L]_B^C = \begin{pmatrix} -t^{l-2} & 0 & -lt^{l-1} \\ 0 & t^l & -lt^{i+l-1} \\ 0 & 0 & t^l \end{pmatrix}$$

Proof. The proof is a straight forward calculation. \square

We see the structure matrix $[L]_B^C$ depend on the integers i, l . Maybe one will get examples with more than two different structures of $\mathcal{O}_{\mathbf{P}}$ -module on the locally free sheaf $\mathcal{O}(l-2) \oplus \mathcal{O}(l) \oplus \mathcal{O}(l)$.

Example 5.6. *Vector bundles on real and complex manifolds.*

Note: If K is the field of complex numbers and X a smooth projective scheme of finite type over K , let $X(\mathbf{R})$ denote the underlying real smooth manifold of X . Let $\mathrm{KO}(X(\mathbf{R}))$ be the Grothendieck group of the category of real smooth finite rank vectorbundles on $X(\mathbf{R})$. There is a canonical map

$$\gamma : \mathrm{K}_0(X) \rightarrow \mathrm{KO}(X(\mathbf{R}))$$

sending the class of a locally free finite rank \mathcal{O}_X -module \mathcal{E} to the class of it's underlying finite rank real vectorbundle $\mathcal{E}(\mathbf{R})$. In many cases the exact sequence

$$0 \rightarrow \mathcal{I} \otimes \mathcal{E} \rightarrow \mathcal{J}_{\mathcal{I}}^1(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow 0$$

is split as sequence of real smooth vectorbundles on $X(\mathbf{R})$ since $X(\mathbf{R})$ is a real compact manifold and all short exact sequences of vector bundles split. It follows there is an isomorphism

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(\mathbf{R})^{\mathrm{left}} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})(\mathbf{R})^{\mathrm{right}}$$

of real smooth vectorbundles. Let

$$c_{\mathcal{I}}(\mathcal{E}) = [\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{\mathrm{left}}] - [\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{\mathrm{right}}] \in \mathrm{K}_0(X).$$

It follows there is an equality

$$\gamma(c_{\mathcal{I}}(\mathcal{E})) = 0$$

in $\mathrm{KO}(X(\mathbf{R}))$.

Example 5.7. *The classical Atiyah sequence.*

Let $\mathcal{E} = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_e)$ be a locally free sheaf on \mathbf{P}^1 of rank e over any field K . Define

$$\deg(\mathcal{E}) = d_1 + \cdots + d_e.$$

Lemma 5.8. *There is an isomorphism*

$$\phi : \mathrm{K}_0(\mathbf{P}^1) \cong \mathbf{Z} \oplus \mathbf{Z}$$

defined by

$$\phi([\mathcal{E}]) = (\deg(\mathcal{E}), rk(\mathcal{E})).$$

Proof. By the projective bundle formula there is an isomorphism

$$\mathrm{K}_0(\mathbf{P}^1) \cong \mathbf{Z}[h]/(1-h)^2$$

of rings. We get

$$\phi([\mathcal{O}(-1)]^d) = h^d = (1+h-1)^d = 1+d(h-1) = 1-dt$$

where t is the class of $1 - h$ in $K_0(\mathbf{P}^1)$. We get similarly

$$\phi([\mathcal{O}(1)]^d) = 1 - d(h - 1) = 1 + d(1 - h) = 1 + dt.$$

It follows

$$\phi([\mathcal{E}]) = 1 + d_1 t + \cdots + 1 + d_e t = rk(\mathcal{E}) + deg(\mathcal{E})t$$

and the claim of the Lemma follows. \square

Let $\mathcal{J}^k(\mathcal{O}(d))$ be the k 'th sheaf of principal parts of $\mathcal{O}(d)$ on \mathbf{P}^1 when $1 \leq k \leq d$. By [10] it follows

$$deg(\mathcal{J}^k(\mathcal{O}(d))^{left}) = deg(\mathcal{J}^k(\mathcal{O}(d))^{right}).$$

Corollary 5.9. *There is an equality*

$$c_\Omega(\mathcal{O}(d)) = 0$$

in $K_0(\mathbf{P}^1)$

Proof. By Lemma 5.8 it follows

$$\begin{aligned} c_\Omega(\mathcal{O}(d)) &= [\mathcal{J}^1(\mathcal{O}(d))^{left}] - [\mathcal{J}^1(\mathcal{O}(d))^{right}] = \\ &= (rk(\mathcal{J}^1(\mathcal{O}(d))^{left}), deg(\mathcal{J}^1(\mathcal{O}(d))^{left})) - \\ &= (rk(\mathcal{J}^1(\mathcal{O}(d))^{right}), deg(\mathcal{J}^1(\mathcal{O}(d))^{right})) = 0. \end{aligned}$$

The Corollary follows. \square

The triviality of the class $c_\Omega(\mathcal{O}(d))$ is related to the fact that the first order jet bundle $\mathcal{J}^1(\mathcal{O}(d))$ is an extension of two abelianized $\mathcal{O}_{\mathbf{P}^1}$ -modules: $\Omega_{\mathbf{P}^1}^1 \otimes \mathcal{O}(d)$ and $\mathcal{O}(d)$: the Atiyah sequence

$$0 \rightarrow \Omega_{\mathbf{P}^1}^1 \otimes \mathcal{O}(d) \rightarrow \mathcal{J}^1(\mathcal{O}(d)) \rightarrow \mathcal{O}(d) \rightarrow 0$$

is an exact sequence of left and right $\mathcal{O}_{\mathbf{P}^1}$ -modules and there are isomorphisms

$$(\Omega_{\mathbf{P}^1}^1 \otimes \mathcal{O}(d))^{left} \cong (\Omega_{\mathbf{P}^1}^1 \otimes \mathcal{O}(d))^{right}$$

and

$$\mathcal{O}(d)^{left} \cong \mathcal{O}(d)^{right}$$

of $\mathcal{O}_{\mathbf{P}^1}$ -modules. Hence

$$\begin{aligned} c_\Omega(\mathcal{O}(d)) &= [\mathcal{J}^1(\mathcal{O}(d))^{left}] - [\mathcal{J}^1(\mathcal{O}(d))^{right}] = \\ &= [(\Omega_{\mathbf{P}^1}^1 \otimes \mathcal{O}(d))^{left}] + [\mathcal{O}(d)^{left}] - [(\Omega_{\mathbf{P}^1}^1 \otimes \mathcal{O}(d))^{right}] - [\mathcal{O}(d)^{right}] = 0 \end{aligned}$$

This argument holds in the following general case:

Theorem 5.10. *There is an equality*

$$c_\Omega(\mathcal{E}) = 0$$

in $K_0(X)$ where X/S is differentially smooth and \mathcal{E} locally free of finite rank.

Proof. The Atiyah sequence is exact as left and right \mathcal{O}_X -modules and there are isomorphisms

$$(\Omega_X^1 \otimes \mathcal{E})^{left} \cong (\Omega_X^1 \otimes \mathcal{E})^{right}$$

of \mathcal{O}_X -modules. \square

This example motivates the construction given in this paper: The left and right \mathcal{O} -module \mathcal{I} must be non-abelianized for the class $c_{\mathcal{I}}(\mathcal{E})$ to be non-trivial in $K_0(X)$. The class $c_{\mathcal{I}}(\mathcal{E})$ lies in an Ext-group which is computable hence it should be easy to check if there is an isomorphism

$$\mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{left} \cong \mathcal{J}_{\mathcal{I}}^1(\mathcal{E})^{right}$$

of \mathcal{O}_X -modules.

There is ongoing work where the characteristic class $c_{\mathcal{I}}(\mathcal{E})$ is studied (see [12]).

REFERENCES

- [1] M. Andre, Homologie des algebres commutatives, *Grundlehren Math. Wiss.* no. 206 (1974)
 - [2] M. Atiyah, Complex analytic connections in fibre bundles, *Trans. AMS* no. 85 (1957)
 - [3] A. Grothendieck, EGA IV Etude locale de schemas et des morphismes de schemas, *Publ. Math. IHES* no. 20 (1964)
 - [4] L. Illusie, Complexe cotangent et deformations I, *Lecture Notes in Math.* Vol. 239 (1971)
 - [5] L. Illusie, Complexe cotangent et deformations II, *Lecture Notes in Math.* Vol. 283 (1972)
 - [6] M. Karoubi, Homologie cyclique et K-theorie, *Asterisque* no 149 (1987)
 - [7] H. Maakestad, A note on the principal parts on projective space and linear representations, *Proc. of the AMS* Vol. 133 no. 2 (2004)
 - [8] H. Maakestad, Chern classes and Lie-Rinehart algebras, *Indagationes Math.* (2008)
 - [9] H. Maakestad, Chern classes and Exan functors, *In progress* (2009)
 - [10] H. Maakestad, Principal parts on the projective line over arbitrary rings, *Manuscripta Math.* 126, no. 4 (2008)
 - [11] H. Maakestad, On parameter spaces of right \mathcal{O}_X -structures, *In progress* (2010)
 - [12] H. Maakestad, On the structure of jetbundles on projective space, *In progress* (2011)
- E-mail address:* h_maakestad@hotmail.com